ON THE EVALUATION OF CERTAIN INTEGRALS WITH REGULAR KERNELS OF THE CAUCHY TYPE

(O VYCHISLENII NEKOTORYKH INTEGRALOV S REGULIARNYM IADROM TIPA KOSHI)

PMM Vol.24, No.6, 1960, pp. 1114-1122

G.N. PYKHTEEV
(Novosibirsk)

(Received April 15, 1960)

In the solution of many problems in hydrodynamics, in the theory of elasticity and in the theory of filtration there occur integrals with regular kernels of the Cauchy type taken along intervals of the real axis. These integrals can be reduced to either one of the following two types:

$$G(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,dt, \quad E(x) = \frac{\sqrt{x^2-1}}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\frac{dt}{\sqrt{1-t^2}} \qquad (1 \leqslant x \leqslant \infty) \qquad (0.1)$$

If the function $f(t)$ satisfies Hölder's [1,2] condition on the interval $[-1, 1]$, then if $x > 1$ these integrals are ordinary regular Riemann integrals. At the point $x = 1$ the integral G can have a singularity of the logarithmic type.

Below we derive formulas which make it possible to obtain approximate expressions for the integrals $G(x)$ and $E(x)$ and to estimate the error in the approximation. The derived formulas contain functions defined in terms of known elementary functions or in terms of rapidly converging series. Tables are given for some of these functions.

1. We introduce into our consideration Chebyshev's polynomials of the first and second kind

$$T_n(t) = \cos n \cos^{-1} t, \qquad U_n(t) = \sin n \cos^{-1} t \qquad (1.1)$$

and also the variable x^{**} defined by the equations

$$x^{**} = x - \sqrt{x^2-1}, \qquad x = (1 + x^{**2}) / 2x^{**} \qquad (1 \leqslant x \leqslant \infty,\ 0 \leqslant x^{**} \leqslant 1) \quad (1.2)$$

It is not difficult to show that $T_n(t)$ and $U_n(t)$ satisfy the relations

$$\frac{1}{\pi}\int_{-1}^{1}\frac{U_n(t)}{t-x}\,dt = -x^{**n}, \qquad \frac{\sqrt{x^2-1}}{\pi}\int_{-1}^{1}\frac{T_n(t)}{t-x}\frac{dt}{\sqrt{1-t^2}} = -x^{**n} \qquad (1.3)$$

Let us assume that the function $f(t)$, which enters into the integrals $G(x)$ and $E(x)$, satisfies Hölder's condition on the interval $[-1, 1]$. Then the integrals $G(x)$ and $E(x)$ can be represented in the form of power series

$$G(x) = -\sum_{n=1}^{\infty} b_n x^{**n}, \qquad E(x) = -\frac{a_0}{2} - \sum_{n=1}^{\infty} a_n x^{**n} \qquad (0 \leqslant x^{**} \leqslant 1) \qquad (1.4)$$

where

$$a_n = \frac{2}{\pi}\int_{-1}^{1} f(t)\,T_n(t)\frac{dt}{\sqrt{1-t^2}}, \qquad b_n = \frac{2}{\pi}\int_{-1}^{1} f(t)\,U_n(t)\frac{dt}{\sqrt{1-t^2}} \qquad (1.5)$$

Indeed, it follows from the theory of Fourier series that the function $f(t)$ in this case can be represented on the interval $[-1, 1]$ as a series in terms of Chebyshev's polynomials of the first kind

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n T_n(t) \qquad (-1 \leqslant t \leqslant 1) \qquad (1.6)$$

or as a series in terms of Chebyshev polynomials of the second kind

$$f(t) = \sum_{n=1}^{\infty} b_n U_n(t) \qquad (-1 \leqslant t \leqslant 1) \qquad (1.7)$$

Let us substitute the series (1.7) for $f(t)$ in the integral $G(x)$ and the series (1.6) for $f(t)$ in the integral $E(x)$. Making use of the relation (1.3) we obtain Formulas (1.4). The use of these formulas for the computation of the integrals $G(x)$ and $E(x)$ is not recommended because the series (1.4) frequently converge quite slowly, while the coefficients a_n and b_n of the arbitrary function $f(t)$ have to be determined by the method of numerical integration. Whenever the sums of the series on the right-hand sides can be found in closed form, then Formulas (1.4) give the exact values of the integrals $G(x)$ and $E(x)$.

2. Let us consider the functions which are defined on the interval $[0, 1]$ by the equations

$$M_1^{(s)}(x) = \sum_{n=2}^{\infty}\frac{1}{(2n-1)^s}x^{2n-1}, \qquad M_2^{(s)}(x) = \sum_{n=2}^{\infty}\frac{1}{(2n)^s}x^{2n} \qquad \begin{pmatrix}0 \leqslant x \leqslant 1\\ s = 1, 2, \ldots\end{pmatrix} \qquad (2.1)$$

$$N_1^{(s)}(x) = \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n-1)^s} x^{2n-1}, \qquad N_2^{(s)}(x) = \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)^s} x^{2n} \qquad \left(\begin{matrix} 0 \leqslant x \leqslant 1 \\ s = 1, 2, \ldots \end{matrix}\right) \quad (2.2)$$

We note the following properties of these functions. The functions $M_1^{(1)}(x)$ and $M_2^{(2)}(x)$ have singularities of the logarithmic type at the point $x = 1$

$$M_1^{(1)}(x) = \frac{1}{2} \ln \frac{1+x}{1-x}, \qquad M_2^{(1)}(x) = -\frac{1}{2} \ln(1-x^2) - \frac{1}{2} x^2$$

The remaining functions are continuous on the interval $[0, 1]$. The functions $N_1^{(1)}(x)$ and $N_2^{(2)}(x)$ can be expressed in the form

$$N_1^{(1)}(x) = x - \tan^{-1} x, \qquad N_2^{(1)}(x) = \frac{1}{2} x^2 - \frac{1}{2} \ln(1 + x^2)$$

In Tables 1 and 2, given in this work, are presented the values of some of the functions $M_1^{(s)}(x)$, $M_2^{(s)}(x)$, $N_1^{(s)}(x)$ and $N_2^{(s)}(x)$ evaluated with accuracy to four places.

TABLE 1.

x	$M_1^{(1)}$	$M_1^{(2)}$	$M_1^{(3)}$	$M_1^{(4)}$	$M_2^{(1)}$	$M_2^{(2)}$	$M_2^{(3)}$	$M_2^{(4)}$
0.1	0.0003	0.0001	$0.0^{2}004$	$0.0^{2}001$	$0.0^{2}002$	$0.0^{2}001$	$0.0^{3}002$	$0.0^{4}004$
0.2	0.0027	0.0009	0.0003	0.0001	0.0004	0.0003	$0.0^{2}002$	$0.0^{2}001$
0.3	0.0100	0.0031	0.0010	0.0003	0.0022	0.0005	0.0001	$0.0^{2}003$
0.4	0.0236	0.0076	0.0025	0.0008	0.0072	0.0017	0.0004	0.0001
0.5	0.0493	0.0153	0.0049	0.0016	0.0188	0.0044	0.0011	0.0003
0.6	0.0932	0.0278	0.0087	0.0028	0.0431	0.0097	0.0023	0.0006
0.7	0.1673	0.0473	0.0144	0.0046	0.0917	0.0196	0.0045	0.0010
0.8	0.2986	0.0773	0.02225	0.0070	0.1908	0.0375	0.0081	0.0019
0.9	0.5722	0.1259	0.0341	0.0102	0.4254	0.0713	0.0143	0.0031
1.0	∞	0.2337	0.0518	0.0147	∞	0.1612	0.0253	0.0051

Making use of Equations (1.2) and (1.3), one can easily show that the functions introduced satisfy the following integral relations:

$$\frac{\sqrt{x^2-1}}{\pi} \int_{-1}^{1} \frac{p_1^{(s)}(t)}{t-x} \frac{dt}{\sqrt{1-t^2}} = -M_1^{(s)}(x^{**})$$

$$\frac{\sqrt{x^2-1}}{\pi} \int_{-1}^{1} \frac{(\operatorname{sign} t)\, q_1^{(s)}(t^*)}{t-x} \frac{dt}{\sqrt{1-t^2}} = N_1^{(s)}(x^{**})$$

$$\frac{\sqrt{x^2-1}}{\pi}\int_{-1}^{1}\frac{p_2^{(s)}(t)}{t-x}\frac{dt}{\sqrt{1-t^2}} = -M_2^{(s)}(x^{**}) \tag{2.3}$$

$$\frac{\sqrt{x^2-1}}{\pi}\int_{-1}^{1}\frac{p_2^{(s)}(t^*)}{t-x}\frac{dt}{\sqrt{1-t^2}} = -N_2^{(s)}(x^{**})$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{p_1^{(s)}(t^*)}{t-x}dt = N_1^{(s)}(x^{**}), \qquad \frac{1}{\pi}\int_{-1}^{1}\frac{q_1^{(s)}(t)}{t-x}dt = -M_1^{(s)}(x^{**}) \tag{2.4}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{q_2^{(s)}(t)}{t-x}dt = -M_2^{(s)}(x^{**}), \qquad \frac{1}{\pi}\int_{-1}^{1}\frac{(\operatorname{sign} t)\, q_2^{(s)}(t^*)}{t-x}dt = N_2^{(s)}(x^{**})$$

where

$$t^* = \sqrt{1-t^2} \tag{2.5}$$

The functions $p_i^{(s)}(t)$ and $q_i^{(s)}$ t which appear in relations (2.3) and (2.4) are defined by the equations

$$p_1^{(s)}(t) = \sum_{n=2}^{\infty}\frac{1}{(2n-1)^s}T_{2n-1}(t), \qquad p_2^{(s)}(t) = \sum_{n=2}^{\infty}\frac{1}{(2n)^s}T_{2n}(t) \quad \begin{pmatrix}-1\leqslant t\leqslant 1\\ s=1,2,\ldots\end{pmatrix} \tag{2.6}$$

$$q_1^{(s)}(t) = \sum_{n=2}^{\infty}\frac{1}{(2n-1)^s}U_{2n-1}(t), \qquad q_2^{(s)}(t) = \sum_{n=2}^{\infty}\frac{1}{(2n)^s}U_{2n}(t) \quad \begin{pmatrix}-1\leqslant t\leqslant 1\\ s=1,2,\ldots\end{pmatrix} \tag{2.7}$$

These functions were introduced in [3] where some of their properties were pointed out. The same work [3] contains tables for these functions which were computed with an accuracy of up to four places.

TABLE 2.

x	$N_1^{(1)}$	$N_1^{(2)}$	$N_1^{(3)}$	$N_1^{(4)}$	$N_2^{(1)}$	$N_2^{(2)}$	$N_2^{(3)}$	$N_2^{(4)}$
0.1	0.0003	0.0001	0.0^2004	0.0^2001	0.0^2002	0.0^2001	0.0^3002	0.0^4004
0.2	0.0026	0.0009	0.0003	0.0001	0.0004	0.0001	0.0^2002	0.0^2001
0.3	0.0085	0.0029	0.0010	0.0003	0.0019	0.0005	0.0001	0.02003
0.4	0.0195	0.0067	0.0023	0.0008	0.0058	0.0015	0.0004	0.0001
0.5	0.0364	0.0128	0.0044	0.0015	0.0134	0.0035	0.0009	0.0002
0.6	0.0596	0.0214	0.0074	0.0026	0.0263	0.0070	0.0018	0.0005
0.7	0.0893	0.0327	0.0116	0.0040	0.0456	0.0124	0.0033	0.0009
0.8	0.1253	0.0469	0.0168	0.0059	0.0726	0.0202	0.0054	0.0014
0.9	0.1672	0.0640	0.0233	0.0082	0.0867	0.0307	0.0084	0.0021
1.0	0.2146	0.0840	0.0310	0.0111	0.1534	0.0444	0.0123	0.0033

3. Next we define the function $f^{(s)}(t)$ by means of the equation

$$f^{(s)}(t) = \left(\frac{d^s}{d\theta^s} f(\cos\theta)\right)_{\theta=\cos^{-1}} \qquad (s = 0, 1, \ldots) \tag{3.1}$$

and call it the trigonometric derivative of the sth order of the function $f(t)$. We denote by $W_\gamma^{(2k)}(M_{2k};\ -1,1)$ the class of functions satisfying the following conditions:

1) Every function $f(t)$ belonging to this class possesses a continuous trigonometric derivative $f^{(s)}(t)$ $(s = 0,\ 1,\ \ldots)$ up to and including the $(2k-1)$ order on the interval $[-1,1]$, except at the point $t = 0$.

2) The trigonometric derivative of order $2k$ of this function satisfies the following inequality:

$$|f^{(2k)}(t)| \leqslant M_{2k} \tag{3.2}$$

3) At the point $t = 0$ the function $f(t)$ and its trigonometric derivative $f^{(s)}(t)$ can have a discontinuity of the first kind, i.e.

$$2\gamma^{(s)} = f^{(s)}(+0) - f^{(s)}(-0) \tag{3.3}$$

The class of functions $W_\gamma^{(2k)}(M_{2k};\ -1,1)$ is a type of generalization of the class $W^{(r)}(M;\ a,\ b)$ considered by Nikol'skii [4,5].

We introduce the following notation:

$$2\gamma_1^{(s)} = f^{(s)}(1) + f^{(s)}(-1), \qquad 2\gamma_2^{(s)} = f^{(s)}(1) - f^{(s)}(-1) \qquad (s = 0, 1, \ldots, 2k) \tag{3.4}$$

$$a_1^* = a_1, \qquad a_2^* = a_2; \qquad b_1^* = b_1, \qquad b_2^* = b_2 \tag{3.5}$$

$$\begin{aligned}
a_{2m}^* &= a_{2m} + \frac{4}{\pi}\sum_{s=1}^{k} \frac{(-1)^s}{(2m)^{2s}} \left((-1)^m \gamma^{(2s-1)} - \gamma_2^{(2s-1)}\right) \\
a_{2m-1}^* &= a_{2m-1} - \frac{4}{\pi}\sum_{s=1}^{k} \frac{(-1)^s}{(2m-1)^{2s}} \left((-1)^m (2m-1)\, \gamma_1^{2(s-1)} + \gamma_1^{(2s-1)}\right) \\
b_{2m}^* &= b_{2m} - \frac{4}{\pi}\sum_{s=1}^{k} \frac{(-1)^s}{(2m)^{2s-1}} \left((-1)^m \gamma_1^{2(s-1)} - \gamma_2^{2(s-1)}\right) \\
b_{2m-1}^* &= b_{2m-1} - \frac{4}{\pi}\sum_{s=1}^{k} \frac{(-1)^s}{(2m-1)^{2s}} \left((-1)^m \gamma_1^{(2s-1)} - (2m-1)\, \gamma_1^{2(s-1)}\right)
\end{aligned} \tag{3.6}$$

where a_n and b_n are Fourier coefficients defined by Formulas (1.5).

We introduce the number $N(n, k)$ defined by the equation

$$N(n, k) = 4(n+1)^{-2k}\left(1 + \ln\frac{\pi}{2} + \frac{1}{2k} + \frac{1}{2n} + \ln n\right) \tag{3.7}$$

In [3] there were given two representations of the function $f(t) \in W_\gamma^{(2k)}(M_{2k}; -1, 1)$ which involved the functions $p_i^{(s)}(t)$ and $q_i^{(s)}(t)$.

Theorem 1. Let

$$f(t) \in W_\gamma^{(2k)}(M_{2k}; -1, 1).$$

Then the following two representations of the function $f(t)$ on the interval $[-1, 1]$ are valid:

$$f(t) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^s\left[\gamma_1^{(2s-1)}p_1^{(2s)}(t) - (\operatorname{sign} t)\,\gamma^{2(s-1)}q_1^{(2s-1)}(t^*) + \right. \tag{3.8}$$

$$\left. + \gamma_2^{(2s-1)}p_2^{(2s)}(t) - \gamma^{(2s-1)}p_2^{(2s)}(t^*)\right] + \frac{a_0}{2} + \sum_{m=1}^{n} a_m^* T_m(t) + r_n^{(1)}(t)$$

$$f(t) = -\frac{4}{\pi}\sum_{s=1}^{k}(-1)^s\left[\gamma^{(2s-1)}p_1^{(2s)}(t^*) + \gamma_1^{2(s-1)}q_1^{(2s-1)}(t) + \right. \tag{3.9}$$

$$\left. + \gamma_2^{2(s-1)}q_2^{(2s-1)}(t) + (\operatorname{sign} t)\,\gamma^{2(s-1)}q_2^{(2s-1)}(t^*)\right] + \sum_{m=1}^{n} b_m^* U_m(t) + r_n^{(2)}(t)$$

where $r_n^{(1)}(t)$ and $r_n^{(2)}(t)$ satisfy the inequalities

$$|r_n^{(1)}(t)| < M_{2k}N(n, k), \quad |r_n^{(2)}(t)| < M_{2k}N(n, k) \tag{3.10}$$

We note that the quantities a_n^* and b_n^* are representable in the form

$$\begin{aligned} a_n^* &= \frac{(-1)^k}{n^{2k}}a_n^{(2k)}, \qquad & a_n^{(2k)} &= \frac{2}{\pi}\int_{-1}^{1} f^{(2k)}(t)\,T_n(t)\frac{dt}{\sqrt{1-t^2}} \\ b_n^* &= \frac{(-1)^k}{n^{2k}}b_n^{(2k)}, \qquad & b_n^{(2k)} &= \frac{2}{\pi}\int_{-1}^{1} f^{(2k)}(t)\,U_n(t)\frac{dt}{\sqrt{1-t^2}} \end{aligned} \tag{3.11}$$

These formulas can be used for defining a_n^* and b_n^* if the derivative $f^{(2k)}(t)$ is known. Next, we construct mth-degree polynomials involving the quantities $a_\nu x^{**\nu}$ and $b_\nu x^{**\nu}$, and establish upper bounds for them.

Theorem 2. Let the function $f(t) \in W_\gamma^{(2k)}(M_{2k}; -1, 1)$. Then the finite sums

$$\sigma_m^{(1)}(x^{**}) = \sum_{\nu=1}^{m} a_\nu^{(2k)} x^{**\nu}, \qquad \sigma_m^{(2)}(x^{**}) = \sum_{\nu=1}^{m} b_\nu^{(2k)} x^{**\nu} \tag{3.12}$$

where $a^{(2k)}$ and $b^{(2k)}$ are defined by (3.11), satisfy the inequalities

$$|\sigma_m^{(s)}(x^{**})| < 4M_{2k}\left(1 + \ln\frac{\pi}{2} + \ln m\right) \qquad (s = 1, 2) \tag{3.13}$$

Proof. Let us substitute in (3.12) in place of $a^{(2k)}$ and $b^{(2k)}$ their expressions from (3.11), and obtain

$$\sigma_m^{(1)}(x^{**}) = \frac{2}{\pi}\int_{-1}^{1} f^{(2k)}(t)\left(\sum_{\nu=1}^{m} x^{**\nu} T_\nu(t)\right)\frac{dt}{\sqrt{1-t^2}}$$
$$\sigma_m^{(2)}(x^{**}) = \frac{2}{\pi}\int_{-1}^{1} f^{(2k)}(t)\left(\sum_{\nu=1}^{m} x^{**\nu} U_\nu(t)\right)\frac{dt}{\sqrt{1-t^2}} \tag{3.14}$$

In the sequel we will assume that $m \geqslant 2$, for if $m = 1$ one can easily see that the inequality (3.13) is satisfied. Indeed

$$|\sigma_1^{(1)}(x^{**})| \leqslant M_{2k}\frac{2}{\pi}\int_{-1}^{1}|T_1(t)|\frac{dt}{\sqrt{1-t^2}} \leqslant 2M_{2k} < 4M_{2k}\left(1 + \ln\frac{\pi}{2}\right)$$

$$|\sigma_1^{(2)}(x^{**})| \leqslant M_{2k}\frac{2}{\pi}\int_{-1}^{1} U_1(t)\frac{dt}{\sqrt{1-t^2}} = M_{2k}\frac{4}{\pi} < 4M_{2k}\left(1 + \ln\frac{\pi}{2}\right)$$

The sums which occur in Formulas (3.14) can be represented in the form

$$\sum_{\nu=1}^{m} x^{**\nu} T_\nu(t) = \frac{(1 - x^{**2})(1 - x^{**m}T_m(t)) - (1 - x^{**}t)(1 - x^{**m-2}T_m(t))}{1 - 2x^{**}t + x^{**2}} -$$
$$- \frac{x^{**m-2}(1 - x^{**2})(1 - x^{**}t)T_m(t) - x^{**m+1}U_m(t)\sqrt{1-t^2}}{1 - 2x^{**}t + x^{**2}} \tag{3.15}$$

$$\sum_{\nu=1}^{m} x^{**\nu} U_\nu(t) = \frac{x^{**}(1 - x^{**m})\sqrt{1-t^2} + x^{**m+1}(1 - T_m(t))\sqrt{1-t^2}}{1 - 2x^{**}t + x^{**2}} -$$
$$- \frac{1}{2}\,\frac{x^{**m}(1 - x^{**2})U_m(t) - x^{**m}(1 - 2x^{**}t + x^{**2})U_m(t)}{1 - 2x^{**}t + x^{**2}} \tag{3.16}$$

Hence, we have

$$\sigma_m^{(1)}(x^{**}) = \frac{2}{\pi}(1 - x^{**2})\int_{-1}^{1} f^{(2k)}(t)\frac{1 - x^{**m}T_m(t)}{1 - 2x^{**}t + x^{**2}}\frac{dt}{\sqrt{1-t^2}} -$$
$$- \frac{2}{\pi}\int_{-1}^{1} f^{(2k)}(t)\frac{(1 - x^{**}t)(1 - x^{**m-2}T_m(t))}{1 - 2x^{**}t + x^{**2}}\frac{dt}{\sqrt{1-t^2}} -$$

$$-\frac{2}{\pi}x^{**m-2}(1-x^{**2})\int_{-1}^{1}f^{(2k)}(t)\frac{(1-x^{**}t)\,T_m(t)}{1-2x^{**}t+x^{**2}}\frac{dt}{\sqrt{1-t^2}}+$$

$$+\frac{2}{\pi}x^{**m+1}\int_{-1}^{1}\frac{f^{(2k)}(t)U_m(t)}{1-2x^{**}t+x^{**2}}dt$$

$$\sigma_m^{(2)}(x^{**})=\frac{2}{\pi}x^{**}(1-x^{**m})\int_{-1}^{1}\frac{f^{(2k)}(t)}{1-2x^{**}t+x^{**2}}dt+$$

$$+\frac{2}{\pi}x^{**m+1}\int_{-1}^{1}f^{(2k)}(t)\frac{1-T_m(t)}{1-2x^{**}t+x^{**2}}dt-$$

$$-\frac{1}{\pi}x^{**m}(1-x^{**2})\int_{-1}^{1}\frac{f^{(2k)}(t)\,U_m(t)}{1-2x^{**}t+x^{**2}}\frac{dt}{\sqrt{1-t^2}}+\frac{1}{\pi}x^{**m}\int_{-1}^{1}f^{(2k)}(t)\,U_m(t)\frac{dt}{\sqrt{1-t^2}}$$

From this it can be easily seen that

$$|\sigma_m^{(1)}(x^{**})|\leqslant M_{2k}\Big[\frac{2}{\pi}(1-x^{**2})\int_{-1}^{1}\frac{1-x^{**}T_m(t)}{1-2x^{**}t+x^{**2}}\frac{dt}{\sqrt{1-t^2}}+$$

$$+\frac{2}{\pi}\int_{-1}^{1}\frac{(1-x^{**}t)(1-x^{**m-2}T_m(t))}{1-2x^{**}t+x^{**2}}\frac{dt}{\sqrt{1-t^2}}+ \tag{3.17}$$

$$+\frac{2}{\pi}(1-x^{**2})\int_{-1}^{1}\frac{(1-x^{**}t)\,|T_m(t)|}{1-2x^{**}t+x^{**2}}\frac{dt}{\sqrt{1-t^2}}+\frac{2}{\pi}x^{**2}\int_{-1}^{1}\frac{|U_m(t)|}{1-2x^{**}t+x^{**2}}dt\Big]$$

$$|\sigma_m^{(2)}(x^{**})|\leqslant M_{2k}\Big[\frac{2}{\pi}x^{**2}\int_{-1}^{1}\frac{1-T_m(t)}{1-2x^{**}t+x^{**2}}dt+\frac{2}{\pi}x^{**}(1-x^{**m})\int_{-1}^{1}\frac{dt}{1-2x^{**}t+x}+$$

$$+\frac{1}{\pi}(1-x^{**2})\int_{-1}^{1}\frac{|U_m(t)|}{1-2x^{**}t+x^{**2}}\frac{dt}{\sqrt{1-t^2}}+\frac{1}{\pi}\int_{-1}^{1}|U_m(t)|\cdot\frac{dt}{\sqrt{1-t^2}}\Big] \tag{3.18}$$

Making use of Formulas (1.1) to (1.5) and of some properties of Chebyshev polynomials $T_n(t)$ and $U_n(t)$, one can show that the following inequalities hold:

$$\frac{2}{\pi}(1-x^{**2})\int_{-1}^{1}\frac{1-x^{**m}T_m(t)}{1-2x^{**}t+x^{**2}}\frac{dt}{\sqrt{1-t^2}}\leqslant 2 \tag{3.19}$$

$$\frac{2}{\pi}\int_{-1}^{1}\frac{(1-x^{**}t)(1-x^{**m-2}T_m(t))}{1-2x^{**}t+x^{**2}}\frac{dt}{\sqrt{1-t^2}}\leqslant 2 \qquad (m\geqslant 2) \tag{3.20}$$

$$\frac{2}{\pi}(1-x^{**2})\int_{-1}^{1}\frac{(1-x^{**}t)\,|T_m(t)|}{1-2x^{**}t+x^{**2}}\frac{dt}{\sqrt{1-t^2}}\leqslant 2 \tag{3.21}$$

$$\frac{2}{\pi}x^{**2}\int_{-1}^{1}\frac{|U_m(t)|}{1-2x^{**}t+x^{**2}}dt < 1+\ln\frac{\pi}{2}+\ln m \tag{3.22}$$

$$\frac{2}{\pi}x^{**}(1-x^{**m})\int_{-1}^{1}\frac{dt}{1-2x^{**}t+x^{**2}}dt < 2\left(1+\ln\frac{\pi}{2}+\ln m\right) \tag{3.23}$$

$$\frac{2}{\pi}x^{**2}\int_{-1}^{1}\frac{1-T_m(t)}{1-2x^{**}t+x^{**2}}dt < 2\left(1+\ln\frac{\pi}{2}+\ln m\right) \tag{3.24}$$

$$\frac{1}{\pi}(1-x^{**2})\int_{-1}^{1}\frac{|U_m(t)|}{1-2x^{**}t+x^{**2}}\frac{dt}{\sqrt{1-t^2}} \leqslant 1,\quad \frac{1}{\pi}\int_{-1}^{1}|U_m(t)|\frac{dt}{\sqrt{1-t^2}} \leqslant 1 \tag{3.25}$$

Strengthening the inequality (3.17) with the aid of (3.19) to (3.22), and the inequality (3.18) by means of the estimates (3.23) to (3.25), we obtain

$$|\sigma_m^{(1)}(x^{**})| < M_{2k}\left(7+\ln\frac{\pi m}{2}\right) < 4M_{2k}\left(1+\ln\frac{\pi}{2}+\ln m\right)$$

$$|\sigma_m^{(2)}(x^{**})| < M_{2k}\left(5+3\ln\frac{\pi m}{2}\right) < 4M_{2k}\left(1+\ln\frac{\pi}{2}+\ln m\right)$$

4. The results of the preceding section make it possible to obtain approximate formulas for the evaluation of the integrals $G(x)$ and $E(x)$.

Theorem 3. Let $f(t)\in W_\gamma^{(2k)}(M_{2k};\ -1,1)$. Then

$$G(x)\approx G_n^{(k)}(x)=\frac{4}{\pi}\sum_{s=1}^{k}(-1)^s\left[\gamma_1^{2(s-1)}M_1^{(2s-1)}(x^{**})+\gamma_2^{2(s-1)}M_2^{(2s-1)}(x^{**})-\right.$$
$$\left.-\gamma^{(2s-1)}N_1^{(2s)}(x^{**})-\gamma^{2(s-1)}N_2^{(2s-1)}(x^{**})\right]-\sum_{m=1}^{n}b_m^{\,*}x^{**m} \tag{4.1}$$

$$E(x)\approx E_n^{(k)}(x)=\frac{4}{\pi}\sum_{s=1}^{k}(-1)^{s-1}\left[\gamma_1^{(2s-1)}M_1^{(2s)}(x^{**})+\gamma_2^{(2s-1)}M_2^{(2s)}(x^{**})+\right.$$
$$\left.+\gamma^{2(s-1)}N_1^{(2s-1)}(x^{**})-\gamma^{(2s-1)}N_2^{(2s)}(x^{**})\right]-\frac{a_0}{2}-\sum_{m=1}^{n}a_m^{\,*}x^{**m} \tag{4.2}$$

whereby

$$|G(x)-G_n^{(k)}(x)| < 2M_{2k}N(n,k),\quad |E(x)-E_n^{(k)}(x)| < 2M_{2k}N(n,k) \tag{4.3}$$

Proof. Let us substitute into the integrals $E(x)$ and $G(x)$ for $f(t)$

Expressions (3.8) and (3.9), respectively. Making use of Equations (1.3), (2.3) and (2.4), we obtain

$$G(x) = G_n^{(k)}(x) + R_n^{(1)}(x), \qquad E(x) = E_n^{(k)}(x) + R_n^{(2)}(x)$$

where

$$R_n^{(1)}(x) = -\sum_{m=n+1}^{\infty} b_m^* x^{**m}, \qquad R_n^{(2)}(x) = -\sum_{m=n+1}^{\infty} a_m^* x^{**m}$$

It remains to prove that $R_n^{(1)}(x)$ and $R_n^{(2)}(x)$ satisfy the inequalities

$$|R_n^{(s)}(x)| < 2M_{2k}N(n, k) \qquad (s = 1, 2)$$

If one introduces into consideration the sums $\sigma_n^{(1)}(x^{**})$ and $\sigma_n^{(2)}(x^{**})$ defined by Equations (3.12), one obtains the following equation for every $R_n^{(s)}(x)$ $(s = 1, 2)$:

$$|R_n^{(s)}(x)| = \left| \sum_{m=n+1}^{\infty} \frac{\sigma_m^{(s)}(x^{**}) - \sigma_{m-1}^{(s)}(x^{**})}{m^{2k}} \right| \qquad (s = 1, 2)$$

$$= \left| -\frac{\sigma_n^{(s)}(x^{**})}{(n+1)^{2k}} + \sum_{m=n+1}^{\infty} \left(\frac{1}{m^{2k}} - \frac{1}{(m+1)^{2k}} \right) \sigma_m^{(s)}(x^{**}) \right|$$

Let us estimate each term of the last series with the aid of (3.13) and the inequalities

$$\ln \frac{m+1}{m} < \frac{1}{m}, \qquad \sum_{m=n+1}^{\infty} \frac{1}{(m+1)^{2k+1}} < \frac{1}{2k(n+1)^{2k}}$$

We obtain the result

$$|R_n^{(s)}(x)| < M_{2k} \frac{4}{(n+1)^{2k}} \left(2 + 2 \ln \frac{\pi n}{2} + \frac{1}{n} + \frac{1}{k} \right) = 2M_{2k} N(n, k)$$

which was to be proved.

We note three special cases when Formulas (4.1) and (4.2) are simplified.

1. The function $f(t)$ and its derivatives are continuous at the point $t = 0$:

$$G(x) \approx G_n^{(k)}(x) = \frac{4}{\pi} \sum_{s=1}^{k} (-1)^s [\gamma_1^{2(s-1)} M_1^{(2s-1)}(x^{**}) + \gamma_2^{2(s-1)} M_2^{(2s-1)}(x^{**})] -$$

$$- \sum_{m=1}^{n} b_m^* x^{**m}, \qquad |G(x) - G_n^{(k)}(x)| < 2M_{2k} N(n, k) \qquad (4.4)$$

$$E(x) \approx E_n^{(k)}(x) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^{s-1}[\gamma_1^{(2s-1)}M_1^{(2s)}(x^{**}) + \gamma_2^{(2s-1)}M_2^{(2s)}(x^{**})] - $$
$$-\frac{a_0}{2} - \sum_{m=1}^{n} a_m^* x^{**m}, \qquad |E(x) - E_n^{(k)}(x)| < 2M_{2k}N(n, k) \tag{4.5}$$

2. The function $f(t)$ is an even function, i.e. $f(-t) = f(t)$:

$$G(x) \approx G_n^{(k)}(x) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^{s}[\gamma_1^{2(s-1)}M_1^{(2s-1)}(x^{**}) - \gamma^{(2s-1)}N_1^{(2s)}(x^{**})] - $$
$$-\sum_{m=1}^{n} b_{2m-1}^* x^{**2m-1}, \qquad |G(x) - G_n^{(k)}(x)| < 2M_{2k}N(2n-1, k) \tag{4.6}$$

$$E(x) \approx E_n^{(k)}(x) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^{s-1}[\gamma_2^{(2s-1)}M_2^{(2s)}(x^{**}) - \gamma^{(2s-1)}N_2^{(2s)}(x^{**})] - $$
$$-\frac{a_0}{2} - \sum_{m=1}^{n} a_{2m}^* x^{**2m}, \qquad |E(x) - E_n^{(k)}(x)| < 2M_{2k}N(2n, k) \tag{4.7}$$

3. The function $f(t)$ is odd, i.e. $f(-t) = -f(t)$:

$$G(x) \approx G_n^{(k)}(x) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^{s}[\gamma_2^{2(s-1)}M_2^{(2s-1)}(x^{**}) - \gamma^{2(s-1)}N_2^{(2s-1)}(x^{**})] - $$
$$-\sum_{m=1}^{n} b_{2m}^* x^{**2m} \qquad |G(x) - G_n^{(k)}(x)| < 2M_{2k}N(2n, k) \tag{4.8}$$

$$E(x) \approx E_n^{(k)}(x) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^{s-1}[\gamma_1^{(2s-1)}M_1^{(2s)}(x^{**}) + \gamma^{2(s-1)}N_1^{(2s-1)}(x^{**})] - $$
$$-\sum_{m=1}^{n} a_{2m-1}^* x^{**2m-1}, \qquad |E(x) - E_n^{(k)}(x)| < 2M_{2k}N(2n-1, k) \tag{4.9}$$

As an example which illustrates the method presented let us consider the integral

$$E(x) = \frac{\sqrt{x^2-1}}{\pi}\int_{-1}^{1}\frac{{}^1/_4\,\pi\,|t|}{t-x}\frac{dt}{\sqrt{1-t^2}} \qquad (1 \leqslant x \leqslant \infty) \tag{4.10}$$

Here the function $f(t) = \pi(t)/4$ is even. Hence, one can use the special case (4.7) and Formulas (4.2) for the evaluation of this integral. Making use of Formulas (3.1) to (3.6), we obtain

$$f^{(2s-1)}(t) = (-1)^s \frac{\pi}{4} (\text{sign } t) \sqrt{1-t^2}, \qquad f^{(2s)}(t) = (-1)^s \frac{\pi}{4} |t|$$

$$\gamma_2^{(2s-1)} = 0, \qquad \gamma^{(2s-1)} = (-1)^s \frac{\pi}{4}, \qquad M_{2k} = \frac{\pi}{4}, \qquad a_0 = 1$$

$$a_{2m} = \frac{(-1)^{m-1}}{4m^2-1} \qquad (m = 1, 2, \ldots), \qquad a_2^* = \frac{1}{3},$$

$$a_{2m}^* = \frac{(-1)^{m-1}}{(2m)^{2k}(4m^2-1)} \qquad (m = 2, 3, \ldots)$$

Substituting the quantities found into (4.7), we obtain

$$E(x) \approx E_n^{(k)}(x) = \sum_{s=1}^{k} N_2^{(2s)}(x^{**}) - \frac{1}{2} - \frac{1}{3} x^{**2} - \sum_{m=2}^{n} \frac{(-1)^{m-1}}{(2m)^{2k}(4m^2-1)} x^{**2m} \qquad (4.11)$$

$$|E(x) - E_n^{(k)}(x)| < \frac{\pi}{2} N(n, k) \qquad (4.12)$$

Formula (4.11) makes it possible to compute with the aid of (4.12) the integral (4.10) within any degree of accuracy. On the other hand, with the aid of Formula (1.4), or directly through integration, one can find the exact value of the integral (4.10) in terms of the elementary functions

$$E(x) = -x \tan^{-1}(x - \sqrt{x^2-1}) \qquad (4.13)$$

Let us compute the approximate values of the considered integral at various points of the interval [1, ∞] and then compare these values with the exact values. We then obtain the following results:

$x =$	1.0	$\cosh\ln\frac{10}{9}$	$\cosh\ln\frac{10}{8}$	$\cosh\ln\frac{10}{7}$	$\cosh\ln\frac{10}{6}$	$\cosh\ln\frac{10}{5}$
$-E(x) =$	0.78540	0.73689	0.69161	0.64999	0.61248	0.57956
$-E_2^{(2)}(x) =$	0.7853	0.7370	0.6916	0.6499	0.6125	0.5796

$x =$	$\cosh\ln\frac{10}{5}$	$\cosh\ln\frac{10}{4}$	$\cosh\ln\frac{10}{3}$	$\cosh\ln\frac{10}{2}$	$\cosh\ln\frac{10}{1}$	∞
$-E(x) =$	0.57956	0.55173	0.52948	0.51323	0.50333	0.5
$-E_2^{(2)}(x) =$	0.5796	0.5517	0.5295	0.5132	0.5033	0.5

This shows that the approximate values of the integral (4.10), computed by means of Formula (4.11) for $n = 2$, $k = 2$, differ from the exact values only by at most 1 at the fourth decimal place. The inequality (4.12) in this case takes the form

$$|E(x) - E_2^{(2)}(x)| < 0.0324$$

that is, Formula (4.11) is actually more exact than is indicated by the inequality (4.12).

In conclusion, the author considers it his duty to thank V.M. Egorov for his help in the preparation of the tables.

BIBLIOGRAPHY

1. Muskhelishvili, N.I., *Singuliarnye integral'nye uravneniia* (*Singular Integral Equations*). Gostekhizdat, 1946.

2. Gakhov, F.D., *Kraevye zadachi* (*Boundary-value Problems*). Fizmatgiz, 1958.

3. Pykhteev, G.N., O vychislenii nekotorykh singuliarnykh integralov s iadrom tipa Koshi (Evaluation of certain singular integrals with a kernel of the Cauchy type). *PMM* Vol. 23, No. 6, 1959.

4. Nikol'skii, S.M., K voprosu ob otsenkakh priblizhenii kvadraturnymi formulami (On the question of estimating approximations with quadrature formulas). *Uspekhi matem. nauk* Vol. 5, No. 2(36), 1950.

5. Nikol'skii, S.M., *Kvadraturnye formuly* (*Quadrature Formulas*). Gostekhizdat, 1958.

Translated by H.P.T.